JOURNAL OF MATHEMATICAL PHYSICS VOLUME 43, NUMBER 1 JANUARY 2002

Character expansions for the orthogonal and symplectic groups

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(Received 6 July 2001; accepted for publication 24 September 2001)

Formulas for the expansion of arbitrary invariant group functions in terms of the characters for the Sp$(2N)$, SO$(2N+1)$, and SO$(2N)$ groups are derived using a combinatorial method. The method is similar to one used by Balantekin to expand group functions over the characters of the U(N) group. All three expansions have been checked for all N by using them to calculate the known expansions of the generating function of the homogeneous symmetric functions. An expansion of the exponential of the traces of group elements, appearing in the finite-volume gauge field partition functions, is worked out for the orthogonal and symplectic groups. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1418014]

I. INTRODUCTION

The expansion of invariant functions of a group into its characters (traces of the representation matrices)[1] is very useful in a number of physical situations. In U(N) lattice gauge theories and in the lattice expansion of the nonlinear U$(N)\times$U(N) sigma model calculation of certain U(N) group integrals are needed.[2–6] If the integrands can be expanded in terms of the U(N) characters, then such integrals can easily be calculated.[3,7,8] Similar U(N) integrals also arise in the statistical theory of nuclear reactions.[9] In 1980 Itzykson and Zuber calculated a particular unitary group integral[10] which turned out to be a special case of a more general formula by Harish-Chandra.[11] The Itzykson–Zuber integral and its generalizations[12–15] are also easily dealt with using character expansions.[1]

The character expansion of an invariant function of group elements is given by

$$f(\det U, TrU, \ldots) = \sum_r a_r \chi_r(U), \tag{1.1}$$

where $\chi_r(U)$ is the character of the representation r. Since group characters form an orthogonal set[16] the coefficients can be obtained by explicitly integrating the product of this function with the characters over the group manifold:

$$a_r = \int dU \chi_r^*(U) f(\det U, TrU, \ldots). \tag{1.2}$$

(Note that a_0 is the integral of the function itself over the group manifold). It is rather difficult to obtain complicated character expansions by explicit integration. In 1984 Balantekin[8] developed a combinatorial method that enabled one to solve for the coefficients in some expansions over the

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U(N) group characters that was quite simple in comparison to performing group integrals. Needing a more general version, the result was recently extended in its range of applicability.[1]

In a parallel development it was shown that the spectral density of the Dirac operator for a gauge theory near its zero eigenvalues should only depend on the symmetries in question.[17–19] Although the original work[17–19] used a Gaussian random matrix model, the results from the random matrix theory can be proven to be universal.[20–24] This implies that the spectral density of the Dirac operator near the origin can be extracted from random matrix theories which provide a description of common aspects of various quantum phenomena (for a review see Ref. 25). Hence to study the low-energy limit of, for example, quantum chromodynamics (QCD), one needs to choose a random matrix theory with the global symmetries of the QCD partition function. The partition functions calculated from the effective field theory and random matrix theory agree.[17,26,27] These random matrix theories are characterized by the Dyson index β which is the number of independent variables per matrix element.[19,28] For fermions in the fundamental representation $\beta=1$ for $N_c=2$ and $\beta=2$ for $N_c\geqslant 2$ where N_c is the number of colors. For fermions in the adjoint representation and $N_c\geqslant 2$ we have $\beta=4$. For $\beta=2$ the low-energy (finite-volume) QCD partition function is the same as the one-link integral of two-dimensional lattice QCD[5,29] and is calculated using the U(N) character expansion[1] and other methods.[30,14,31] For $\beta=4$ the zero momentum Goldstone modes belong to the coset space $\mathrm{SU}(N_f)/\mathrm{SO}(N_f)$[30,15] where N_f is the number of fermion flavors. Hence the finite-volume partition function is a group integral where the argument is the exponential of an $\mathrm{SO}(N_f)$ group element. Similarly in the $\beta=1$ case the coset space of the Goldstone modes is $\mathrm{SU}(2N_f)/\mathrm{Sp}(2N_f)$.[30,15] Massive partition functions of random matrix ensembles with $\beta=1$ and 4 were considered in Refs. 24 and 32. To calculate these partition functions it is very useful to have expressions for character expansions over the orthogonal and symplectic groups. Explicit expressions for these partition functions, for example, may help in finding solutions of Virasoro constraints which were found so far only for the $\beta=2$ case. (For the application of Virasoro constraints on the effective finite volume partition function, see, for example, Refs. 33–35).

The present work is an extension of Balantekin's method of finding the expansion coefficients for expansions over the characters for the symplectic group $\mathrm{Sp}(2N)$, the odd dimensional special orthogonal group $\mathrm{SO}(2N+1)$, and the even dimensional special orthogonal group $\mathrm{SO}(2N)$. Some background material will be treated in Sec. II, including general information about the characters of the groups in question. The procedures for developing the expansions for the different groups are similar, so Sec. III will treat the general idea. The specific expressions will be derived in Sec. IV for $\mathrm{Sp}(2N)$, in Sec. V for $\mathrm{SO}(2N+1)$, and in Sec. VI for $\mathrm{SO}(2N)$. Finally, some examples of expansions will be given for each of the groups in Sec. VII.

For quick reference, the expansions and the most general expressions for their coefficients for $\mathrm{Sp}(2N)$, $\mathrm{SO}(2N+1)$, and $\mathrm{SO}(2N)$ are found in Eqs. (4.10) and (4.11), (5.8) and (5.9), and (6.10) and (6.12), respectively.

II. BACKGROUND AND FORMULAS FOR CHARACTERS

In order to calculate expressions for the character expansions, we will need expressions for the characters. These characters have been furnished by Weyl.[36] For reference, the Weyl formulas for the U(N), $\mathrm{Sp}(2N)$, $\mathrm{SO}(2N+1,\mathfrak{R})$, and $\mathrm{SO}(2N,\mathfrak{R})$ group characters are reprinted below. [We have included the formula for the U(N) group characters for completeness even though we will not need them in the present work.]

In the following, $\det[\mathcal{A}_{ij}]$ refers to the determinant of the $N\times N$ matrix $\mathcal{A}$ whose entry in the ith row and jth column is $\mathcal{A}_{ij}$. Furthermore, we will denote a matrix in the unitary group as U, the symplectic group as P, and the orthogonal group as R, and the eigenvalues of any of these matrices are labeled by t_i. For the U(N) case, there are N eigenvalues that are all phases. For the $\mathrm{Sp}(2N)$ and $\mathrm{SO}(2N)$ cases, there are $2N$ eigenvalues, but they come in pairs of a phase and its reciprocal. Thus, a complete list of eigenvalues would be $t_1,t_2,\ldots,t_N,t_1^{-1},t_2^{-1},\ldots,t_N^{-1}$. For the $\mathrm{SO}(2N+1)$ case, it is the same as the even dimensional cases with an additional eigenvalue of

$t=1$. The determinants given below, then, are determinants of $N\times N$ matrices which contain functions of only the individual eigenvalues, not their reciprocals. Finally, the character is a function of the representation, which is labeled by a partition $(n_1,n_2,\ldots,n_N)$ where the non-negative integers n_i satisfy $n_1\geqslant n_2\geqslant\cdots\geqslant n_N$. Each representation corresponds to a permissible Young tableau.

The expressions for the simple characters of the $\mathrm{U}(N)$, $\mathrm{Sp}(2N)$, and $\mathrm{SO}(2N+1)$ groups are

$$\chi_{(n_1,n_2,\ldots,n_N)}(U)=\frac{\det[t_i^{n_j+N-j}]}{\det[t_i^{N-j}]}, \tag{2.1}$$

$$\chi_{(n_1,n_2,\ldots,n_N)}(P)=\frac{\det[t_i^{n_j+N-j+1}-t_i^{-(n_j+N-j+1)}]}{\det[t_i^{N+1-j}-t_i^{-(N+1-j)}]}, \tag{2.2}$$

and

$$\chi_{(n_1,n_2,\ldots,n_N)}(R)=\frac{\det[t_i^{n_j+N-j+1/2}-t_i^{-(n_j+N-j+1/2)}]}{\det[t_i^{N+1/2-j}-t_i^{-(N+1/2-j)}]}, \tag{2.3}$$

respectively.

The $\mathrm{SO}(2N)$ case requires more attention. We define

$$\mathcal{C}_{(n_1,n_2,\ldots,n_N)}(R)=\frac{\det[t_i^{n_j+N-j}+t_i^{-(n_j+N-j)}-\delta_{jN}\delta_{n_N0}]}{\det[t_i^{N-j}+t_i^{-(N-j)}-\delta_{jN}]} \tag{2.4}$$

and

$$\mathcal{S}_{(n_1,n_2,\ldots,n_N)}(R)=\frac{\det[t_i^{n_j+N-j}-t_i^{-(n_j+N-j)}]}{\det[t_i^{N-j}+t_i^{-(N-j)}-\delta_{jN}]}. \tag{2.5}$$

(The notation in the previous two equations is nonstandard, but they give the proper elements as stated in Ref. 16 in a more modern and manipulable form.) $\mathcal{C}_{(n_1,n_2,\ldots,n_N)}(R)$ alone is the simple character of $\mathrm{SO}(2N)$ if and only if $n_N=0$. If $n_N\neq 0$, then $\mathcal{C}_{(n_1,n_2,\ldots,n_N)}(R)$ is a double character. For this case, the simple characters are given by $\frac{1}{2}(\mathcal{C}_{(n_1,n_2,\ldots,n_N)}(R)\pm\mathcal{S}_{(n_1,n_2,\ldots,n_N)}(R))$. In the present work, only the expression for $\mathcal{C}_{(n_1,n_2,\ldots,n_N)}(R)$ given in Eq. (2.4) will be needed. This statement will be justified in Sec. VI where $\mathrm{SO}(2N)$ is treated.

One last property of these expressions that will be useful for checking the reliability of the expansions derived in this article is the value of the characters for representations corresponding to Young tableaux of one row ($n_1=n$, all others are 0) and one column ($n_n=1$ for all n up to some value, all others are 0). The characters for representations with one row, labeled (n), and one column, labeled (1^n), are

$$\chi_{(n)}(U)=h_n(t_i),\quad \chi_{(1^n)}(U)=a_n(t_i), \tag{2.6}$$

$$\chi_{(n)}(P)=h_n(t_i,t_i^{-1}),\quad \chi_{(1^n)}(P)=a_n(t_i,t_i^{-1})-a_{n-2}(t_i,t_i^{-1}), \tag{2.7}$$

$$\chi_{(n)}(R)=h_n(t_i,t_i^{-1},1)-h_{n-2}(t_i,t_i^{-1},1),\quad \chi_{(1^n)}(R)=a_n(t_i,t_i^{-1},1), \tag{2.8}$$

$$\mathcal{C}_{(n)}(R)=h_n(t_i,t_i^{-1})-h_{n-2}(t_i,t_i^{-1}),\quad \mathcal{C}_{(1^n)}(R)=a_n(t_i,t_i^{-1}), \tag{2.9}$$

for the $\mathrm{U}(N)$, $\mathrm{Sp}(2N)$, $\mathrm{SO}(2N+1)$, and $\mathrm{SO}(2N)$ groups, respectively. The functions $h_n(t_i)$ are the homogeneous symmetric functions of order n, and the functions $a_n(t_i)$ are the elementary symmetric functions of order n. Further discussion is given in Ref. 1.

III. GENERAL PROPERTIES OF THE DERIVATION

We are now ready to derive the form for the expansions of group functions over the characters of the various groups mentioned in the previous section. As with any expansion, the crux of this issue is being able to determine and calculate the coefficients in the expansion. The goal of the next four sections will be to find these coefficients.

The derivation is very similar to the one used by Balantekin[1] in finding the coefficients of the expansion over the unitary group characters. As the expressions for the group characters for $\mathrm{Sp}(2N)$, $\mathrm{SO}(2N+1)$, and $\mathrm{SO}(2N)$ are all similar, the derivations for all three will proceed in much the same manner. To make the general method more transparent, the common aspects of the derivation will be presented in this section without mention of the specific groups. The following three sections will be devoted to using the result of this section to derive expressions for the coefficients in the expansions over the characters in each group.

We begin by noting that each of the expressions for the characters given by Eqs. (2.2)–(2.4) are all ratios of determinants, so that we can write

$$\chi_{(n_1,n_2,\ldots,n_N)}(M)=\frac{\mathcal{N}}{\mathcal{D}}, \tag{3.1}$$

where M is a matrix element of the group in question and $\mathcal{N}$ and $\mathcal{D}$ refer to numerator and denominator. For any of these groups, the denominator $\mathcal{D}$ can be expressed generally as

$$\mathcal{D}=\det[t_i^{N-j+q}\pm t_i^{-(N-j+q)}-\delta_{q0}\delta_{jN}], \tag{3.2}$$

where q can take on the value 1 for the $\mathrm{Sp}(2N)$ group, $\frac{1}{2}$ for the $\mathrm{SO}(2N+1)$ group, and 0 for the $\mathrm{SO}(2N)$ group. In this form, we choose the appropriate value of q and the proper sign of the $\pm$ sign to specify which group we are discussing. Namely, we see that the minus sign will be used for $\mathrm{Sp}(2N)$ and $\mathrm{SO}(2N+1)$ whereas the plus sign will be used for $\mathrm{SO}(2N)$.

In following the derivation of the $\mathrm{U}(N)$ expansion given in Ref. 1, we define a "generating function," $G(x,t)$, to be some function of a variable t and any necessary parameters x. Later, we will take t to be an eigenvalue of a group matrix. For now, we expand the generating function in a power series in the variable t around $t=0$. Thus,

$$G(x,t)=\sum_{n=-\infty}^{\infty}A_n(x)t^n. \tag{3.3}$$

We assume that the series expansion converges for $|t|=1$. However, there are no other restrictions on the coefficients, so that some of the $A_n(x)$ can be zero. For instance, if the expansion is a Taylor series, then $A_n\equiv 0$ for all $n<0$.

Now, we define a function $\mathcal{F}$ by

$$\mathcal{F}=\mathcal{D}\prod_{i=1}^{N}G(x,t_i)G(x,t_i^{-1}), \tag{3.4}$$

where $\mathcal{D}$ is given in Eq. (3.2). By using the definition of $G(x,t)$ from Eq. (3.3), $\mathcal{F}$ can be written as (suppressing the x dependence of A_n)

$$\mathcal{F}=\mathcal{D}\left[\sum_{n=-\infty}^{\infty}A_nt_1^n\right]\left[\sum_{n=-\infty}^{\infty}A_nt_1^{-n}\right]\left[\sum_{n=-\infty}^{\infty}A_nt_2^n\right]\left[\sum_{n=-\infty}^{\infty}A_nt_2^{-n}\right]\cdots\left[\sum_{n=-\infty}^{\infty}A_nt_N^n\right]\left[\sum_{n=-\infty}^{\infty}A_nt_N^{-n}\right], \tag{3.5}$$

or, by combining the product of sums over the same variable, it can be written as

$$\mathcal{F}=\mathcal{D}\left[\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}A_nA_pt_1^{n-p}\right]\left[\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}A_nA_pt_2^{n-p}\right]\cdots\left[\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}A_nA_pt_N^{n-p}\right]. \tag{3.6}$$

To proceed, we use the expression for $\mathcal{D}$ in Eq. (3.2). This determinant can be laboriously expanded as an alternating sum of products of the elements [see Eq. (A3)]. Upon doing so, we can combine the factors of the variable t_i in the determinant with the factor in Eq. (3.6) of the same variable. However, before naively doing so, we notice that there is a symmetry in the exponents in the determinant. We also notice that the double summations of the t_i's are unaffected by interchange of the dummy indices n and p. So the symmetry of the exponents will be preserved if we use $n-p$ in the product of the first term, $p-n$ in the product of the second term, and split the delta term in half using $n-p$ in the first one and $p-n$ in the second one. Upon doing so, we find that the new expression is again a determinant. Rewriting this as a determinant, we obtain

$$\mathcal{F}=\det\left[\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}A_nA_p\left(t_i^{N-j+q+n-p}\pm t_i^{-(N-j+q+n-p)}-\frac{1}{2}\delta_{q0}\delta_{jN}(t_i^{n-p}+t_i^{p-n})\right)\right]. \tag{3.7}$$

Now, we change variables, defining a new integer $r=N-j+n-p$. This gives

$$\mathcal{F}=\det\left[\sum_{p=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}A_{r-N+j+p}A_p\left(t_i^{r+q}\pm t_i^{-(r+q)}-\frac{1}{2}\delta_{q0}\delta_{jN}(t_i^{r-N+j}+t_i^{N-j-r})\right)\right]. \tag{3.8}$$

The order of the summation for r and p is interchangeable. Also, the delta term chooses only $N=j$. Thus,

$$\mathcal{F}=\det\left[\sum_{r=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}A_{r-N+j+p}A_p\left(t_i^{r+q}\pm t_i^{-(r+q)}-\frac{1}{2}\delta_{q0}\delta_{jN}(t_i^{r}+t_i^{-r})\right)\right]. \tag{3.9}$$

We notice that all of the dependence on the dummy variable p can be isolated by defining

$$c_{r,j}=\sum_{p=-\infty}^{\infty}A_{r-N+j+p}A_p\,. \tag{3.10}$$

Also, by combining the delta term with the other term [again remembering that it is only present for the SO$(2N)$ case in which $q=0$ and we use the $+$ sign], we can write our expression for $\mathcal{F}$ as

$$\mathcal{F}=\det\left[\sum_{r=-\infty}^{\infty}c_{r,j}(t_i^{r+q}\pm t_i^{-(r+q)})\left(1-\frac{1}{2}\delta_{q0}\delta_{jN}\right)\right]. \tag{3.11}$$

We have come to the point in the derivation where it will be necessary to specialize Eq. (3.11) for the three different types of groups by making appropriate choices for q and the $\pm$ sign. This will be taken up in the next three sections.

IV. THE EXPANSION OVER SP$(2N)$ CHARACTERS

We begin with the Sp$(2N)$ case, as it is the simplest one. The starting point will be Eq. (3.11). For the Sp$(2N)$ case, $q=1$ and we choose the minus sign. Thus, we have

$$\mathcal{F}=\det\left[\sum_{r=-\infty}^{\infty}c_{r,j}(t_i^{r+1}-t_i^{-(r+1)})\right], \tag{4.1}$$

where $c_{r,j}$ is defined in Eq. (3.10). Before proceeding, it is beneficial to change dummy indices again by letting $r+1\rightarrow r$. This gives us

$$\mathcal{F}=\det\left[\sum_{r=-\infty}^{\infty} c'_{r,j}(t_i^r-t_i^{-r})\right], \tag{4.2}$$

where

$$c'_{r,j}=\sum_{p=-\infty}^{\infty} A_{r-N+j-1+p}A_p . \tag{4.3}$$

This sum over r from $-\infty$ to ∞ can be broken up into positive r, negative r, and $r=0$. The $r=0$ term vanishes because $t_i^0-t_i^0=0$. Then, changing the negative values to positive by replacing r with $-r$ and collecting terms, we get

$$\mathcal{F}=\det\left[\sum_{r=0}^{\infty} d_{r,j}(t_i^r-t_i^{-r})\right], \tag{4.4}$$

where

$$d_{r,j}=c'_{r,j}-c'_{-r,j} . \tag{4.5}$$

Equation (4.4) is very similar to an expression that is treated in Theorem 1.2.1 from Ref. 37. We will need a slightly more general form of this theorem, which we present in the Appendix. Using the result, Eq. (A8), we get

$$\mathcal{F}=\sum_{r_1>r_2>\cdots>r_N\geqslant 0}^{\infty} \det[d_{r_j,i}]\det[t_i^{r_j}-t_i^{-r_j}]. \tag{4.6}$$

Now, if, in the summation, $r_N=0$, then both determinants vanish, so we can restrict $r_N\geqslant 1$. Let us define

$$r_j=n_j+N-j+1. \tag{4.7}$$

Then, $r_j>r_{j+1}$ implies that $n_j\geqslant n_{j+1}$. Furthermore, since $r_N\geqslant 1$, then $n_N\geqslant 0$. Thus, we can write the summation as

$$\mathcal{F}=\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty} \det[d_{n_j+N-j+1,i}]\det[t_i^{n_j+N-j+1}-t_i^{-(n_j+N-j+1)}]. \tag{4.8}$$

In the above expression, the second determinant is seen to be exactly the numerator in the Weyl formula for the characters of the symplectic group given in Eq. (2.2). We even have the appropriate restrictions on the values of the n_i that are necessary to make the equation valid. Thus, we can write the above expression as

$$\mathcal{F}=\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty} \det[d_{n_j+N-j+1,i}]\mathcal{N}. \tag{4.9}$$

Now we recall our definition of $\mathcal{F}$ from Eq. (3.4). If we divide both sides by the denominator $\mathcal{D}$ and recall that our expression for the character of the Sp$(2N)$ group is $\chi_{(n_1,n_2,\ldots,n_N)}(P)=\mathcal{N}/\mathcal{D}$, then we obtain

$$\prod_{i=1}^{N} G(x,t_i)G(x,t_i^{-1})=\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty} \det[d_{n_j+N-j+1,i}]\chi_{(n_1,n_2,\ldots,n_N)}(P). \tag{4.10}$$

This is our desired character expansion over the $\mathrm{Sp}(2N)$ group! It is a sum over all irreducible representations of the symplectic group. Expressions for the coefficients are obtained using Eqs. (4.5) and (4.3). The result is

$$d_{n_j+N-j+1,i}=\sum_{p=-\infty}^{\infty} A_p(A_{n_j+i-j+p}-A_{-n_j-2N-2+i+j+p}). \tag{4.11}$$

In the special case in which the expansion of the generating function $G(x,t)$ is a Taylor series expansion with $A_p\equiv 0$ for all $p<0$, this simplifies slightly to

$$d_{n_j+N-j+1,i}=\sum_{p=0}^{\infty} A_p(A_{p+|n_j+i-j|}-A_{p+n_j+2N+2-i-j}). \tag{4.12}$$

We defer examples of the usage of this expansion until Sec. VII.

V. THE EXPANSION OVER SO$(2N+1)$ CHARACTERS

Once again, we start from Eq. (3.11). For $\mathrm{SO}(2N+1)$, we have $q=\frac{1}{2}$ and we choose the minus sign. Then, we have

$$\mathcal{F}=\det\left[\sum_{r=-\infty}^{\infty} c_{r,j}(t_i^{r+1/2}-t_i^{-(r+1/2)})\right] \tag{5.1}$$

and $c_{r,j}$ is defined in Eq. (3.10).

This sum over r from $-\infty$ to ∞ can be broken up into ranges of $r\geqslant 0$ and $r<0$, which gives

$$\mathcal{F}=\det\left[\sum_{r=0}^{\infty} c_{r,j}(t_i^{r+1/2}-t_i^{-(r+1/2)})+\sum_{r=-\infty}^{-1} c_{r,j}(t_i^{r+1/2}-t_i^{-(r+1/2)})\right]. \tag{5.2}$$

Changing variables in the second summation using $r\rightarrow-(r+1)$ and collecting terms, we get

$$\mathcal{F}=\det\left[\sum_{r=0}^{\infty} d_{r,j}(t_i^{r+1/2}-t_i^{-(r+1/2)})\right], \tag{5.3}$$

where

$$d_{r,j}=c_{r,j}-c_{-r-1,j}\,. \tag{5.4}$$

Once again, we refer to Eq. (A8) in the Appendix to simplify Eq. (5.3) and we write

$$\mathcal{F}=\sum_{r_1>r_2>\cdots>r_N\geqslant 0}^{\infty} \det[d_{r_j,i}]\det[t_i^{r_j+1/2}-t_i^{-(r_j+1/2)}]. \tag{5.5}$$

If we define

$$r_j=n_j+N-j, \tag{5.6}$$

then the summation becomes

$$\mathcal{F}=\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty} \det[d_{n_j+N-j,i}]\det[t_i^{n_j+N-j+1/2}-t_i^{-(n_j+N-j+1/2)}]. \tag{5.7}$$

The second determinant is simply the numerator of the Weyl formula for $\mathrm{SO}(2N+1)$ as given in Eqs. (2.3), so using the definition of $\mathcal{F}$ from Eq. (3.4) and dividing by $\mathcal{D}$ gives

$$\prod_{i=1}^{N} G(x,t_i)G(x,t_i^{-1}) = \sum_{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N \geqslant 0}^{\infty} \det[d_{n_j+N-j,i}]\chi_{(n_1,n_2,\ldots,n_N)}(R). \tag{5.8}$$

This is the expansion for the SO$(2N+1)$ group, again a sum over all irreducible representations. Note, however, that this expression does not include the spinor representations of SO$(2N+1)$. The expression for the coefficient is found using Eqs. (5.4) and (3.10) with the result given by

$$d_{n_j+N-j,i} = \sum_{p=-\infty}^{\infty} A_p(A_{n_j+i-j+p} - A_{-n_j-2N-1+i+j+p}). \tag{5.9}$$

In the special case of a Taylor series with $A_p \equiv 0$ for all $p<0$, this simplifies to

$$d_{n_j+N-j,i} = \sum_{p=0}^{\infty} A_p(A_{p+|n_j+i-j|} - A_{p+n_j+2N+1-i-j}). \tag{5.10}$$

We conclude this section with a reminder that care must be taken in the usage of the above formulas for SO$(2N+1)$. One must remember that the number 1 is always an additional eigenvalue of the matrix R. Thus, in forming group functions, one must manually include a factor of $G(x,1)$ on both sides of the equation in order to have a function on the left hand side that treats all eigenvalues equally. This tricky point will be illustrated by example in Sec. VII, after we treat the SO$(2N)$ case in the next section.

VI. THE EXPANSION OVER SO$(2N)$ CHARACTERS

One more time, we start from Eq. (3.11). Recall that for SO$(2N)$, we have $q=0$ and we use the $+$ sign. Thus, we have

$$\mathcal{F} = \det\left[\sum_{r=-\infty}^{\infty} c_{r,j}(t_i^r + t_i^{-r})\left(1 - \frac{1}{2}\delta_{jN}\right)\right], \tag{6.1}$$

where $c_{r,j}$ is defined in Eq. (3.10). The delta function term serves to divide each entry in the last column by a factor of 2. When taking the determinant, a factor of 2 comes out and divides the equation. Thus,

$$\mathcal{F} = \frac{1}{2}\det\left[\sum_{r=-\infty}^{\infty} c_{r,j}(t_i^r + t_i^{-r})\right]. \tag{6.2}$$

This sum over r from $-\infty$ to ∞ can be broken up into positive r, negative r, and $r=0$. Then changing the negative values to positive by replacing r with $-r$ and collecting terms, we get

$$\mathcal{F} = \frac{1}{2}\det\left[\sum_{r=0}^{\infty} d_{r,j}\left(1 - \frac{1}{2}\delta_{r0}\right)(t_i^r + t_i^{-r})\right], \tag{6.3}$$

where

$$d_{r,j} = c_{r,j} + c_{-r,j} \tag{6.4}$$

and the δ_{r0} is inserted to ensure the correct coefficient for $r=0$. Once again, we can use Eq. (A8) from the Appendix to simplify Eq. (6.3) which gives

$$\mathcal{F} = \frac{1}{2}\sum_{r_1 > r_2 > \cdots > r_N \geqslant 0}^{\infty} \det[d_{r_j,i}]\det\left[(t_i^{r_j} + t_i^{-r_j})\left(1 - \frac{1}{2}\delta_{r_j 0}\right)\right]. \tag{6.5}$$

Let us define

$$r_j = n_j + N - j. \tag{6.6}$$

Then the summation becomes

$$\mathcal{F} = \frac{1}{2} \sum_{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N \geqslant 0}^{\infty} \det[d_{n_j+N-j,i}] \det\left[(t_i^{n_j+N-j} + t_i^{-(n_j+N-j)}) \left(1 - \frac{1}{2} \delta_{n_j+N-j,0} \right) \right]. \tag{6.7}$$

Focusing on the second determinant on the right hand side, we can multiply the two binomials to give

$$\det\left[t_i^{n_j+N-j} + t_i^{-(n_j+N-j)} - \frac{1}{2} \delta_{n_j+N-j,0} (t_i^{n_j+N-j} + t_i^{-(n_j+N-j)}) \right]. \tag{6.8}$$

Now, the delta function is only nonzero when $n_j + N - j = 0$, which can only occur for $j = N$ and $n_N = 0$ because n_j is non-negative. In this event, the exponents vanish and the sum in parentheses becomes 2, which cancels the $\frac{1}{2}$. Thus, we can write Eq. (6.7) as

$$\mathcal{F} = \frac{1}{2} \sum_{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N \geqslant 0}^{\infty} \det[d_{n_j+N-j,i}] \det[t_i^{n_j+N-j} + t_i^{-(n_j+N-j)} - \delta_{jN} \delta_{n_N 0}]. \tag{6.9}$$

We see that the second determinant is precisely the appropriate numerator in the Weyl formula for the quantity $\mathcal{C}_{(n_1, n_2, \ldots, n_N)}(R)$ given in Eq. (2.4). By recalling the definition of $\mathcal{F}$ from Eq. (3.4) and dividing by $\mathcal{D}$, we get

$$\prod_{i=1}^{N} G(x, t_i) G(x, t_i^{-1}) = \sum_{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N \geqslant 0}^{\infty} \frac{1}{2} \det[d_{n_j+N-j,i}] \mathcal{C}_{(n_1, n_2, \ldots, n_N)}(R), \tag{6.10}$$

This is the character expansion for SO$(2N)$. As with the SO$(2N+1)$ case, the expansion does not include the spinor representations. Note that the above expansion is not an expansion over the simple characters of the SO$(2N)$ group because the $\mathcal{C}$'s are double characters if $n_N > 0$ as discussed in Sec. II. If one desires an expansion over the simple characters, one can write

$$\mathcal{C} = \tfrac{1}{2}(\mathcal{C} + \mathcal{S}) + \tfrac{1}{2}(\mathcal{C} - \mathcal{S}), \tag{6.11}$$

which puts the two simple characters on the right hand side, as explained earlier. At the present time, we find it simpler to apply the formula in the state that it is in. The expression for the coefficient is found using Eqs. (6.4) and (3.10) and is found to be

$$d_{n_j+N-j,i} = \sum_{p=-\infty}^{\infty} A_p (A_{n_j+i-j+p} + A_{-n_j-2N+i+j+p}). \tag{6.12}$$

In the special case of a Taylor series with $A_p = 0$ for all $p < 0$, this simplifies to

$$d_{n_j+N-j,i} = \sum_{p=0}^{\infty} A_p (A_{p+|n_j+i-j|} + A_{p+n_j+2N-i-j}). \tag{6.13}$$

The derivations of the expansions are complete. We now turn to some examples.

VII. EXAMPLES OF CHARACTER EXPANSIONS

In this section, we give some examples of expansions of group functions over the characters of the Sp$(2N)$, SO$(2N+1)$, and SO$(2N)$ groups. In Sec. VII A, we will present the expansion of the generating function of the homogeneous symmetric functions. This can be used as a check of the formulas derived in the present article, as the expansions are known. In Sec. VII B, we present the expansion for the function exp($xTrM$), where M is some matrix element of one of the three groups we treat.

A. Homogeneous symmetric functions

Consider the generating function of the homogeneous symmetric functions, namely

$$G(x,t)=\frac{1}{1-xt}=\sum_{n=0}^{\infty} x^n t^n. \tag{7.1}$$

Thus, we have $A_n(x)=x^n$ for $n\geqslant 0$ and $A_n(x)=0$ otherwise. Consider the $\mathrm{Sp}(2N)$ expansion. Note that

$$\prod_{i=1}^{N} G(x,t_i)G(x,t_i^{-1})=\frac{1}{\det[I-xP]}, \tag{7.2}$$

where I is the $2N\times 2N$ identity matrix. The expansion is given by Eq. (4.10). Since the series expansion of the generating function in Eq. (7.1) does not contain negative powers of t, the coefficients are given by Eq. (4.12). Thus, the coefficients are given by

$$\det[d_{n_j+N-j+1,i}]=\det\left[\sum_{p=0}^{\infty} x^p(x^{p+|n_j+i-j|}-x^{p+n_j+2N+2-i-j})\right], \tag{7.3}$$

which after simplifying becomes

$$\det[d_{n_j+N-j+1,i}]=\det\left[\frac{x^{|n_j+i-j|}-x^{n_j+2N+2-i-j}}{1-x^2}\right]. \tag{7.4}$$

We simplify by noticing that if $n_2\geqslant 1$, then the first column of the determinant is a multiple of the second column, thereby making the determinant vanish. Thus, n_2 must be zero in order to have a nonvanishing coefficient. Now, since $n_2\geqslant n_3$ and so on, we see that the only surviving terms in the expansion are those for which $n_2=n_3=\cdots=n_N=0$. This corresponds to one row Young tableaux, labeled (n) earlier. The above determinant then becomes

$$\det[d_{n_j+N-j+1,i}]=\det\left[\frac{x^{|n_1\delta_{1j}+i-j|}-x^{n_1\delta_{1j}+2N+2-i-j}}{1-x^2}\right]. \tag{7.5}$$

This, in turn, can be written as

$$\det[d_{n_j+N-j+1,i}]=\frac{x^{n_1}}{(1-x^2)^N}\det[x^{|i-j|}-x^{2N+2-i-j}]. \tag{7.6}$$

By an induction argument on the dimension of the determinant on the right hand side, one can prove that

$$\det[x^{|i-j|}-x^{2N+2-i-j}]=(1-x^2)^N \tag{7.7}$$

and thus

$$\det[d_{n_j+N-j+1,i}]=x^{n_1}, \tag{7.8}$$

where we recall that all n's other than n_1 are 0. Then, the character expansion is Eq. (4.10) with the coefficients found above is

$$\frac{1}{\det[I-xP]}=\prod_{i=1}^{N}\left(\frac{1}{1-xt_i}\right)\left(\frac{1}{1-xt_i^{-1}}\right)=\sum_{n_1=0}^{\infty}x^{n_1}\chi_{(n_1)}(P). \tag{7.9}$$

If we use Eq. (2.7), which relates the character of one row Young tableaux to the homogeneous symmetric functions, we have

$$\frac{1}{\det[I-xP]}=\prod_{i=1}^{N}\left(\frac{1}{1-xt_i}\right)\left(\frac{1}{1-xt_i^{-1}}\right)=\sum_{n=0}^{\infty}x^{n}h_n(t_i,t_i^{-1}). \tag{7.10}$$

However, this is exactly the defining equation for the homogeneous symmetric functions. Thus, we see that the expansion derived for $\mathrm{Sp}(2N)$ agrees with the known expansion.

To perform the same expansion over the $\mathrm{SO}(2N+1)$ group, one must use caution. As alluded to earlier, we must manually include the eigenvalue 1. Mathematically, we have

$$\frac{1}{\det[I-xR]}=G(x,1)\prod_{i=1}^{N}G(x,t_i)G(x,t_i^{-1}), \tag{7.11}$$

where $G(x,t)$ is still given by Eq. (7.1) and I is the $(2N+1)\times(2N+1)$ identity matrix. Then, we have the expansion from Eq. (5.8) and we scale both sides by $G(x,1)=(1-x)^{-1}$ to get

$$G(x,1)\prod_{i=1}^{N}G(x,t_i)G(x,t_i^{-1})=\frac{1}{1-x}\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty}\det[d_{n_j+N-j,i}]\chi_{(n_1,n_2,\ldots,n_N)}(R). \tag{7.12}$$

The coefficient is given by Eq. (5.10)

$$\det[d_{n_j+N-j,i}]=\det\left[\sum_{p=0}^{\infty}x^{p}(x^{p+|n_j+i-j|}-x^{p+n_j+2N+1-i-j})\right], \tag{7.13}$$

or, after simplifying,

$$\det[d_{n_j+N-j,i}]=\det\left[\frac{x^{|n_j+i-j|}-x^{n_j+2N+1-i-j}}{1-x^2}\right]. \tag{7.14}$$

Once again it can be shown that if $n_2\geqslant 1$, the first two columns are multiples and the coefficient vanishes. It can also be shown using the result of the similar expression for the $\mathrm{Sp}(2N)$ expansion that if $n_1=n$ and all other $n_i=0$, then

$$\det[d_{n_j+N-j,i}]=\frac{x^n}{1+x} \tag{7.15}$$

so that

$$\frac{1}{\det[I-xR]}=\frac{1}{(1-x)}\prod_{i=1}^{N}\frac{1}{(1-xt_i)}\frac{1}{(1-xt_i^{-1})}=\sum_{n=0}^{\infty}\frac{x^n}{1-x^2}\chi_{(n)}(R). \tag{7.16}$$

Finally, using the value of the single row characters for the $\mathrm{SO}(2N+1)$ group from Eq. (2.8), one can show that

$$\frac{1}{\det[I-xR]}=\frac{1}{(1-x)}\prod_{i=1}^{N}\frac{1}{(1-xt_i)}\frac{1}{(1-xt_i^{-1})}=\sum_{n=0}^{\infty}x^{n}h_n(t_i,t_i^{-1},1), \tag{7.17}$$

which again agrees with the definition of the homogeneous symmetric functions.

We continue on with the same expansion for the SO$(2N)$ group. Using the same generating function and using the character expansion from Eq. (6.10), we have

$$\frac{1}{\det[I-xR]}=\prod_{i=1}^{N}\frac{1}{(1-xt_i)}\frac{1}{(1-xt_i^{-1})}=\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty}\frac{1}{2}\det[d_{n_j+N-j,i}]\mathcal{C}_{(n_1,n_2,\ldots,n_N)}(R). \tag{7.18}$$

Note the appearance of the factor of $\frac{1}{2}$ in this expression. The coefficient is given by Eq. (6.13) to be

$$\det[d_{n_j+N-j,i}]=\det\left[\sum_{p=0}^{\infty}x^p(x^{p+|n_j+i-j|}+x^{p+n_j+2N-i-j})\right], \tag{7.19}$$

or, after simplification,

$$\det[d_{n_j+N-j,i}]=\det\left[\frac{x^{|n_j+i-j|}+x^{n_j+2N-i-j}}{1-x^2}\right]. \tag{7.20}$$

Once again, the first two columns are multiples if $n_2\geqslant 1$, so the only surviving coefficients are the ones corresponding to $n_1=n$, all others are zero. Again, the determinant can be evaluated with the help of the result from the Sp$(2N)$ case, and the result is

$$\det[d_{n_j+N-j,i}]=\frac{2x^n}{1-x^2}. \tag{7.21}$$

We note that the 2 cancels with the $\frac{1}{2}$ built into the SO$(2N)$ expansion and the remaining expression is exactly the same as the SO$(2N+1)$ expression. Thus, we see that the expansions derived in the present work indeed give the correct expansion. In all three of these examples, we have tacitly assumed that N is at least 2, but one can check that the expansions are correct for the $N=1$ cases as well.

We could also consider the generating function for the alternating symmetric functions, $G(x,t)=1-xt$, and calculate the expansions as another check for reliability. One can check that the expansions derived from the present work in fact give the known expansions. This task will not be undertaken in the present work.

B. Expansion of exp$(x\,Tr\,\mathbf{M})$

Now that we have confidence in the character expansions derived here, we can start considering more interesting examples. Of course, any generating function can be chosen as long as it can be expanded in a power series. For our example, we will choose the exponential function because it is expected that this technique will prove useful in performing group integrals that arise in low-energy effective QCD partition functions and the integrals are of exponential functions.

We begin by defining the generating function

$$G(x,t)=e^{xt}=\sum_{n=0}^{\infty}\frac{x^n}{n!}t^n \tag{7.22}$$

so that $A_n(x)=x^n/n!$ for $n\geqslant 0$ and zero otherwise. Let us first consider Sp$(2N)$. We have

$$\prod_{i=1}^{N}G(x,t_i)G(x,t_i^{-1})=e^{x(t_1+t_2+\cdots+t_N+t_1^{-1}+t_2^{-1}+\cdots+t_N^{-1})}=\exp(xTrP). \tag{7.23}$$

Our character expansion is given by Eq. (4.10) as

$$\exp(xTrP)=\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty}\det[d_{n_j+N-j+1,i}]\chi_{(n_1,n_2,\ldots,n_N)}(P),\tag{7.24}$$

where the coefficients are given directly by Eq. (4.12) as

$$\det[d_{n_j+N-j+1,i}]=\det\left[\sum_{p=0}^{\infty}\frac{x^p}{p!}\left(\frac{x^{p+|n_j+i-j|}}{(p+|n_j+i+j|)!}-\frac{x^{p+n_j+2N+2-i-j}}{(p+n_j+2N+2-i-j)!}\right)\right].\tag{7.25}$$

This can be recognized as a modified Bessel function, which has the expansion

$$I_\lambda(x)=\sum_{p=0}^{\infty}\frac{1}{p!(p+\lambda)!}\left(\frac{x}{2}\right)^{2p+\lambda}.\tag{7.26}$$

Note also that $I_\lambda(x)=I_{-\lambda}(x)$ for any x. Thus, we can rewrite the coefficient as (dropping the absolute value sign)

$$\det[d_{n_j+N-j+1,i}]=\det[I_{n_j+i-j}(2x)-I_{n_j+2N+2-i-j}(2x)]\tag{7.27}$$

so that, finally,

$$\exp(xTrP)=\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty}\det[I_{n_j+i-j}(2x)-I_{n_j+2N+2-i-j}(2x)]\chi_{(n_1,n_2,\ldots,n_N)}(P).\tag{7.28}$$

We proceed with the same expansion for the SO$(2N+1)$ group. Using the same generating function, Eq. (5.8) gives us the expansion as

$$\prod_{i=1}^{N}\exp(xt_i)\exp(xt_i^{-1})=\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty}\det[d_{n_j+N-j,i}]\chi_{(n_1,n_2,\ldots,n_N)}(R),\tag{7.29}$$

where Eq. (5.10) gives

$$\det[d_{n_j+N-j,i}]=\det\left[\sum_{p=0}^{\infty}\frac{x^p}{p!}\left(\frac{x^{p+|n_j+i-j|}}{(p+|n_j+i-j|)!}-\frac{x^{p+n_j+2N+1-i-j}}{(p+n_j+2N+1-i-j)!}\right)\right],\tag{7.30}$$

or, using the definition of the modified Bessel function,

$$\det[d_{n_j+N-j,i}]=\det[I_{n_j+i-j}(2x)-I_{n_j+2N+1-i-j}(2x)].\tag{7.31}$$

Thus, our expression is

$$\prod_{i=1}^{N}\exp(xt_i)\exp(xt_i^{-1})=\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty}\det[I_{n_j+i-j}(2x)-I_{n_j+2N+1-i-j}(2x)]\chi_{(n_1,n_2,\ldots,n_N)}(R).\tag{7.32}$$

Now, the left hand side is not yet exp($xTrR$). We need to include the eigenvalue 1. Thus we multiply both sides by e^x and we get

$$\exp(xTrR)=\exp(x)\prod_{i=1}^{N}\exp(xt_i)\exp(xt_i^{-1})$$
$$=e^x\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty}\det[I_{n_j+i-j}(2x)-I_{n_j+2N+1-i-j}(2x)]\chi_{(n_1,n_2,\ldots,n_N)}(R),\tag{7.33}$$

which is the desired expansion. We emphasize the appearance of the e^x on the right hand side of the expression. This extra term is unique to the $SO(2N+1)$ group.

As our final example, we develop the same expansion for the $SO(2N)$ group. As before, we write the expansion from Eq. (6.10) as

$$\exp(xTrR)=\prod_{i=1}^{N}\exp(xt_i)\exp(xt_i^{-1})=\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N>0}^{\infty}\frac{1}{2}\det[d_{n_j+N-j,i}]\mathcal{C}_{(n_1,n_2,\ldots,n_N)}(R).\tag{7.34}$$

The coefficients are given by Eq. (6.13) as

$$\det[d_{n_j+N-j,i}]=\det\left[\sum_{p=0}^{\infty}\frac{x^p}{p!}\left(\frac{x^{p+|n_j+i-j|}}{(p+|n_j+i-j|)!}+\frac{x^{p+n_j+2N-i-j}}{(p+n_j+2N-i-j)!}\right)\right].\tag{7.35}$$

Again using the modified Bessel equation expansion, we get

$$\det[d_{n_j+N-j,i}]=\det[I_{n_j+i-j}(2x)+I_{n_j+2N-i-j}(2x)].\tag{7.36}$$

Thus, the desired expansion is

$$\exp(xTrR)=\sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0}^{\infty}\frac{1}{2}\det[I_{n_j+i-j}(2x)+I_{n_j+2N-i-j}(2x)]\mathcal{C}_{(n_1,n_2,\ldots,n_N)}(R).\tag{7.37}$$

Once again, we emphasize the factor of $\frac{1}{2}$ in this expression. This is unique to the $SO(2N)$ expansion.

VIII. CONCLUSIONS

The present article, along with Refs. 1 and 8 completes the program of finding character expansions for all classical Lie groups. We expect these formulas to be useful in a wide range of applications. We already described some of these applications in the Introduction.

One should emphasize that the success in understanding the relationship between the random matrix theories and the low-lying eigenvalues of the QCD Dirac operator suggests investigating other aspects of QCD in a statistical framework (for a recent review see Ref. 38). More recently a similarity between disordered systems in condensed matter physics and QCD, namely the existence of a universal energy scale known as Thouless energy, was suggested.[39–41] This problem can be treated using the supersymmetry approach.[42,25,43] In the supersymmetry approach to this problem one needs to calculate integrals over supergroups.[44,45] One should note that integration over unitary supergroups was already considered in Refs. 13 and 45–47. Invariant integration over an $Osp(N/2M)$ manifold was also previously discussed in Refs. 48 and 49. An approach based on Gelfand–Tzetlin coordinates was developed and a recursion formula for both ordinary and supergroup integrals was found.[50–53] Character expansions for supergroups may be useful to understand the nature and extent of this approach. The characters of supergroups are given by formulas similar to the Weyl formulas except that complete symmetric functions are replaced by the graded homogeneous symmetric functions or alternately traces by supertraces.[54–57] Since our character expansion formulas are basically combinatorial in nature they are applicable to the supergroups as well by the appropriate substitution of traces with supertraces. Thus one can obtain character expansions of the orthosymplectic supergroup $\mathrm{Osp}(N/2M)$ from our formulas for $\mathrm{SO}(N)$[54] and of the supergroup $\mathrm{P}(N)$ from our formulas for $\mathrm{Sp}(2N)$.[55]

ACKNOWLEDGMENTS

We thank Gernot Akemann for useful discussions. This work was supported in part by the U.S. National Science Foundation Grant No. PHY-0070161 at the University of Wisconsin, in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the Alexander von Humboldt-Stiftung. ABB is grateful to the Max-Planck-Institut für Kernphysik and H. A. Weidenmüller for the very kind hospitality.

APPENDIX: A THEOREM ON DETERMINANTS

Here we take up the issue of slightly generalizing a theorem on determinants that can be found in Hua's book.[37] His Theorem 1.2.1 states that

$$\det\left[\sum_{r=0}^{\infty} d_{r,j} t_i^r\right] = \sum_{r_1>r_2>\ldots>r_N\geqslant 0}^{\infty} \det[d_{r_j,i}]\det[t_i^{r_j}]. \tag{A1}$$

We would like to prove the following more general statement, which we state as a theorem.

Theorem 1: *Let $f_r(t)$ be an arbitrary function of the variable t with dependence on the index r. Then the following equality holds*:

$$\det\left[\sum_{r=0}^{\infty} d_{r,j} f_r(t_i)\right] = \sum_{r_1>r_2>\ldots>r_N\geqslant 0}^{\infty} \det[d_{r_j,i}]\det[f_{r_j}(t_i)]. \tag{A2}$$

To prove the theorem, we recall the expansion of determinants of $N\times N$ matrices, namely,

$$\det[\mathcal{A}_{i,j}] = \sum_{m_1=1}^{N}\sum_{m_2=1}^{N}\cdots\sum_{m_N=1}^{N} \epsilon_{m_1m_2\cdots m_N}\mathcal{A}_{1,m_1}\mathcal{A}_{2,m_2}\cdots\mathcal{A}_{N,m_N}, \tag{A3}$$

where the tensor $\epsilon_{m_1m_2\cdots m_N}$ is completely antisymmetric. Then, the left hand side of Eq. (A2) becomes

$$\det\left[\sum_{r=0}^{\infty} d_{r,j} f_r(t_i)\right] = \sum_{m_1=1}^{N}\sum_{m_2=1}^{N}\cdots\sum_{m_N=1}^{N} \epsilon_{m_1m_2\cdots m_N}\left[\sum_{r_1=0}^{\infty} d_{r_1,m_1} f_{r_1}(t_1)\right] \times\left[\sum_{r_2=0}^{\infty} d_{r_2,m_2} f_{r_2}(t_2)\right]\cdots\left[\sum_{r_N=0}^{\infty} d_{r_N,m_N} f_{r_N}(t_N)\right]. \tag{A4}$$

Isolating the dependence on the m_i's, we get

$$\det\left[\sum_{r=0}^{\infty} d_{r,j} f_r(t_i)\right] = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_N=0}^{\infty} f_{r_1}(t_1)f_{r_2}(t_2)\cdots f_{r_N}(t_N) \times\left[\sum_{m_1=1}^{N}\sum_{m_2=1}^{N}\cdots\sum_{m_N=1}^{N} \epsilon_{m_1m_2\cdots m_N} d_{r_1,m_1}d_{r_2,m_2}\cdots d_{r_N,m_N}\right]. \tag{A5}$$

We recognize the term on the right hand side in the large brackets as a determinant, so

$$\det\left[\sum_{r=0}^{\infty} d_{r,j} f_r(t_i)\right] = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_N=0}^{\infty} f_{r_1}(t_1) f_{r_2}(t_2) \cdots f_{r_N}(t_N) \det[d_{r_i,j}]. \tag{A6}$$

Now, if any of the r_i are equal, then the determinant on the right hand side will vanish because two rows would be identical. Thus, the sum can be restricted to distinct values of the r_i's. Next, since the r_i's are all different, we would like to order them in descending order so that $r_i > r_{i+1}$. In doing so, we would like to not change the form of the determinant on the right hand side. So, for any switch of labels, we permute the rows to leave the form unchanged. This brings in a factor of $+1$ or -1, depending on how many permutations are needed. We can express this simply by using the Nth rank alternating tensor as

$$\det\left[\sum_{r=0}^{\infty} d_{r,j} f_r(t_i)\right] = \sum_{r_1>r_2>\cdots>r_N\geqslant 0}^{\infty} \det[d_{r_i,j}] \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \cdots \sum_{m_N=1}^{N} \epsilon_{m_1 m_2 \cdots m_N} f_{r_{m_1}}(t_1) f_{r_{m_2}}(t_2) \cdots f_{r_{m_N}}(t_N). \tag{A7}$$

Here we note that the term involving m_i sums is a determinant. Thus, taking the transpose of the determinant of the d's, we conclude

$$\det\left[\sum_{r=0}^{\infty} d_{r,j} f_r(t_i)\right] = \sum_{r_1>r_2>\cdots>r_N\geqslant 0}^{\infty} \det[d_{r_j,i}] \det[f_{r_j}(t_i)], \tag{A8}$$

which proves the theorem.

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